Linear–Quadratic optimal control for a class of stochastic Volterra equations : solvability and approximation

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Joint work with
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Motivation

Basic linear-quadratic (LQ) regulator problem with BM noise W:

$$X_t^\alpha = \int_0^t \alpha_s ds + W_t,$$

and a quadratic cost functional on finite horizon T to minimize

$$J(\alpha) = \mathbb{E}\Big[\int_0^T \big(|X_t^\alpha|^2 + \alpha_t^2\big) dt\Big].$$

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Optimal control:

$$\alpha_t^* = -\Gamma_{t,T} X_t^{\alpha^*}, \quad 0 \leq t \leq T,$$

where Γ is a deterministic nonnegative function:

$$\Gamma_{t,T} = tanh(T-t),$$

that is solution to the Riccati equation:

$$\dot{\Gamma}_{t,T} = -1 + \Gamma_{t,T}^2, \quad \Gamma_{T,T} = 0,$$

and thus the associated optimal state process X^{α^*} is a mean-reverting Markov process.

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$$X_t^\alpha = \int_0^t \alpha_s ds + W_t, \quad t \geq 0,$$

replace W by a Gaussian process with memory, typically a fractional Brownian motion

$$X_t^\alpha = \int_0^t \alpha_s ds + \int_0^t (t-s)^{H-1/2} dW_s, \quad t \geq 0,$$

or more generally by stochastic Volterra equations:

$$X_t^\alpha = g_0(t) + \int_0^t K(t-s) b(s, X_s^\alpha, \alpha_s) ds + \int_0^t K(t-s) \sigma(s, X_s^\alpha, \alpha_s) dW_s,$$

Question: how is the structure of the solution modified? Numerics?
Sticking points: stochastic calculus for semimartingales and usual methods for Markov processes no longer available!

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Literature review

Several techniques in the literature for control of stochastic Volterra equations (or fractional Brownian motion):

- Malliavin calculus Agram & Oksendal (2015)
- Gaussian calculus: Duncan & Duncan (2012)
- Backward stochastic Volterra equations: Yong (2006), Wang (2018)
- Path dependent HJB: Han & Wong (2019)

Challenges and Limitations:

1. Difficulty in dealing with fractional Brownian motion with $H \in (0, 1/2)$,
2. Control in the volatility,
3. Lack of numerical methods,
4. **In the LQ framework:** underlying LQ structure not well identified.

Aim: Treat all 4 challenges in one go.

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Set -up

Controlled process in $\mathbb{R}^d$:

$$X_t^\alpha = g_0(t) + \int_0^t K(t-s) b(s, X_s^\alpha, \alpha_s) ds + \int_0^t K(t-s) \sigma(s, X_s^\alpha, \alpha_s) dW_s,$$

where $\alpha_t \in \mathbb{R}^m$ belongs to some admissible set $\mathcal{A}$, $K \in L^2([0,T], \mathbb{R}^{d \times d'})$ and

$$b(t,x,a) = \beta(t) + Bx + Ca, \quad \sigma(t,x,a) = \gamma(t) + Dx + Fa,$$

some matrices B, C, D, F with suitable dimension.

Cost functional:

$$J(\alpha) = \mathbb{E}\left[\int_0^T \left((X_s^\alpha)^\top Q X_s^\alpha + \alpha_s^\top N \alpha_s\right) ds\right]$$

Optimization problem:

$$V_0 = \inf_{\alpha \in \mathcal{A}} J(\alpha)$$

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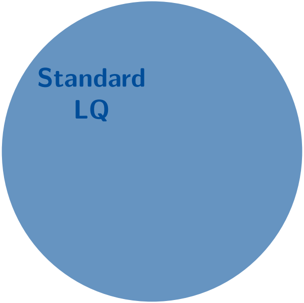
Conventional LQ



- $K \equiv I_d$
- Revisit conventional LQ problems
- From dimension 1, to $\mathbb{R}^d$ to Hilbert spaces.

Deriving the solution

Dimension $d = 1$:

$$dX_s^\alpha = (BX_s^\alpha + C\alpha_s)\,ds + (DX_s^\alpha + F\alpha_s)\,dW_s$$

$$J(\alpha) = \mathbb{E}\left[\int_0^T \left(Q(X_s^\alpha)^2 + N\alpha_s^2\right) ds\right]$$

Ansatz for value function:

$$V_t^\alpha = \Gamma_t X_t^2$$

for some deterministic function $t \to \Gamma_t$ to be determined such that $\Gamma_T = 0$.

Strategy: Inspired by martingale verification argument: Find Γ such that

$$S_t^\alpha := V_t^\alpha + \int_0^t \left(QX_s^2 + N\alpha_s^2\right) ds$$

is a submartingale for every $\alpha \in \mathcal{A}$ and a martingale for $\alpha^\star$.

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By Itô:

$$dS_t^\alpha = X_t^2 \left(\dot{\Gamma}_t + Q + 2B\Gamma_t + D^2\Gamma_t\right) dt$$
$$+ \left(\alpha_t^2(N + F^2\Gamma_t) + 2\alpha_t X_t \left(C\Gamma_t + DF\Gamma_t\right)\right) dt$$
$$+ 2\left(D\Gamma_t X_t^2 + F\alpha_t X_t\right) dW_t$$

Completion of squares: on red term

$$(\bullet) = \left(N + F^2\Gamma_t\right)\left(\alpha_t - \alpha_t^*\right)^2 - \left(N + F^2\Gamma_t\right)^{-1}\left(C\Gamma_t + DF\Gamma_t\right)^2 X_t^2$$

with

$$\alpha_t^* = -\left(N + F^2\Gamma_t\right)^{-1}\left(C\Gamma_t + DF\Gamma_t\right) X_t$$

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Vanishing first term if Γ solves the Backward Riccati equation:

$$\dot{\Gamma}_t = -Q - 2B\Gamma_t - D^2\Gamma_t + \left(N + F^2\Gamma_t\right)^{-1}\left(C\Gamma_t + DF\Gamma_t\right)^2, \quad \Gamma_T = 0.$$

$\Rightarrow M^\alpha = S^\alpha - \int_0^\cdot \left(N + F^2\Gamma_s\right)(\alpha_s - \alpha_s^*)^2 ds$ is a local martingale.
True martingale if

$$\sup_{t \le T} \mathbb{E}\left[X_t^4\right] < \infty.$$

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Deriving the solution

Writing the martingale property $\mathbb{E}\left[M_T^\alpha | \mathcal{F}_t\right] = M_t^\alpha$ we obtain

$$J_t(\alpha) - V_t^\alpha = \mathbb{E}\left[\int_t^T \underbrace{\left(N + F^2\Gamma_s\right)}_{\text{provided } \geq 0} (\alpha_s - \alpha_s^*)^2 ds \Big| \mathcal{F}_t\right] \geq 0,$$

where

$$J_t(\alpha) := \mathbb{E}\left[\int_t^T \left(QX_s^2 + \alpha_s^2\right) ds \Big| \mathcal{F}_t\right].$$

This shows that α^* is an optimal control and $V_t^{\alpha^*}$ is the value function of the problem:

$$V_t^{\alpha^*} = \inf_{\alpha \in \mathcal{A}_t(\alpha^*)} J_t(\alpha)$$

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Dimension 1

$$\mathcal{A} = \left\{ \alpha : \Omega \times [0, T] \to \mathbb{R} \text{ progressive such that } \sup_{0 \le t \le T} \mathbb{E}\left[|\alpha_t|^4 \right] < \infty \right\}$$

Verification result in dimension 1

Assume that

1. There exists a nonnegative solution Γ to the Riccati equation:
2. There exists an admissible control α^* satisfying

$$\alpha_t^* = -\left(N + F^2\Gamma_t\right)^{-1}\left(C\Gamma_t + DF\Gamma_t\right) X_t^{\alpha^*}$$

Then, α^* is an optimal control and $V_t^{\alpha^*} = \Gamma_t (X_t^{\alpha^*})^2$ is the value function of the problem:

$$V_t^{\alpha^*} = \inf_{\alpha} J_t(\alpha)$$

1 and 2 are obtained if

$$Q \ge 0 \quad \text{and} \quad N > 0.$$

Hilbert space

The result also hold for X with values in some Hilbert space $\mathcal{H}$:

$$dX_t^\alpha = (AX_t^\alpha + BX_t^\alpha + C\alpha_t)\,ds + (DX_t^\alpha + F\alpha_t)\,dW_t$$

provided the matrix Riccati equation is replaced by an operator Riccati equation $\boldsymbol{\Gamma}$

$$\begin{aligned}\dot{\boldsymbol{\Gamma}}_t &= -\boldsymbol{\Gamma}_t A - A^*\boldsymbol{\Gamma}_t - Q - B^*\boldsymbol{\Gamma}_t - \boldsymbol{\Gamma}_t B - D^*\boldsymbol{\Gamma}_t D \\ &\quad + (C^*\boldsymbol{\Gamma}_t + F^*\boldsymbol{\Gamma}_t D)(N + F^*\boldsymbol{\Gamma}_t F)^{-1}(C^*\Gamma_t + F^*\boldsymbol{\Gamma}_t D), \quad \boldsymbol{\Gamma}_T = 0.\end{aligned}$$

with a corresponding value function:

$$\begin{aligned}V_t^{\alpha^*} &= \langle X_t^{\alpha^*}, \boldsymbol{\Gamma}_t X_t^{\alpha^*}\rangle_{\mathcal{H}} \\ \alpha_t^* &= -(N + F^*\boldsymbol{\Gamma}_t F)^{-1}(C^*\boldsymbol{\Gamma}_t + F^*\boldsymbol{\Gamma}_t D)\,X_t^{\alpha^*}\end{aligned}$$

See Da Prato (1984), Flandoli (1986).

Solvability of LQ Volterra

$$X_t^\alpha = g_0(t) + \int_0^t K(t-s) b(s, X_s^\alpha, \alpha_s) ds + \int_0^t K(t-s) \sigma(s, X_s^\alpha, \alpha_s) dW_s$$

- Non-Markovian/ non-semimartingale

Lift the process to recover Markovianity:

- Every process X can be made Markovian in infinite-dimension by keeping track of its past $\mathcal{X}_t = (X_s)_{s \le t}$,
- Alternative way: forward lift

$$g_t(s) = \mathbb{E}\left[X_s - \int_t^s K(s-u) b_u du \middle| \mathcal{F}_t \right]$$

(A.J. & El Euch '19, Cuchiero & Teichmann '18, Han & Wong '19, Viens & Zhang '19)

Assumption on K:

Assumption on K: Laplace transform of a $d \times d'$-matrix signed measure μ:

$$K(t) = \int_{\mathbb{R}_+} e^{-\theta t} \mu(d\theta), \quad t > 0,$$

such that

$$\int_{\mathbb{R}_+} \left(1 \wedge \theta^{-1/2}\right) |\mu|(d\theta) < \infty,$$

where $|\mu|$ is the total variation of the measure μ.
Remark: $\mu_{ij}(\mathbb{R}_+)$ not necessarily finite, ie singularity of the kernel at 0 allowed! But $K \in L^2([0,T], \mathbb{R}^{d \times d'})$

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Examples

- $$K(t) = \sum_{i=1}^{n} c_i^n e^{-\theta_i^n t} \quad \mu(d\theta) = \sum_{i=1}^{n} c_i^n \delta_{\theta_i^n}(d\theta)$$
- Fractional kernel $(d = d' = 1)$
  $$K_H(t) = \frac{t^{H-1/2}}{\Gamma(H+1/2)}, \quad \mu_H(d\theta) = \frac{\theta^{-H-1/2}}{\Gamma(H+1/2)\Gamma(1/2-H)}.$$
- Completely monotone kernels K, i.e. K is infinitely differentiable on $(0, \infty)$ such that $(-1)^n K^{(n)}(t)$ is nonnegative for each $t > 0$, (Bernstein's theorem)
- Sums and products...

Markovian representation of X^α

Markovian representation exploiting the structure of the kernel:

- First introduced in Carmona, Coutin & Montseny '00 for the Markovian representation of fractional Brownian motion,
- Recently generalized to uncontrolled stochastic Volterra: A.J. & El Euch '19, Cuchiero & Teichmann '18, Harms & Stefanovits '19.

Markovian representation of X^α

Assumption : $K(t) = \int_{\mathbb{R}_+} e^{-\theta t} \mu(d\theta)$

$$X_t^\alpha = g_0(t) + \int_0^t K(t-s) \underbrace{\left(b(s, X_s^\alpha, \alpha_s)ds + \sigma(s, X_s^\alpha, \alpha_s)dW_s\right)}_{dZ_s^\alpha}$$

$$= g_0(t) + \int_{\mathbb{R}_+} \mu(d\theta) \int_0^t e^{-\theta(t-s)} dZ_s^\alpha$$

$$= g_0(t) + \int_{\mathbb{R}_+} \mu(d\theta) Y_t^\alpha(\theta)$$

where $Y_t^\alpha(\theta) := \int_0^t e^{-\theta(t-s)} dZ_s^\alpha$, $\theta \in \mathbb{R}_+$. In particular, $(Y_t^\alpha)_{t \geq 0}$ is the mild solution of

$$dY_t(\theta) = \left(-\theta Y_t^\alpha(\theta) + b\left(t, g_0(t) + \int_{\mathbb{R}_+} \mu(d\theta') Y_t^\alpha(\theta'), \alpha_t\right)\right) dt$$
$$+ \sigma\left(t, g_0(t) + \int_{\mathbb{R}_+} \mu(d\theta') Y_t^\alpha(\theta'), \alpha_t\right) dW_t, \quad Y_0^\alpha(\theta) = 0.$$

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Markovian representation of X^α

$\Rightarrow$ Markovian problem in $L^1(\mu)$ on the state variables Y^α:
Define the **mean-reverting operator** A^{mr} acting on measurable functions $\varphi \in L^1(\mu)$ by

$$(A^{mr}\varphi)(\theta) = -\theta\varphi(\theta), \quad \theta \in \mathbb{R}_+,$$

and consider the **dual pairing**

$$\langle \varphi, \psi \rangle_\mu = \int_{\mathbb{R}_+} \varphi(\theta)^\top \mu(d\theta)^\top \psi(\theta), \quad (\varphi, \psi) \in L^1(\mu) \times L^\infty(\mu^\top).$$

For any matrix–valued kernel G, we denote by $\boldsymbol{G}$ the **integral operator** induced by G, defined by:

$$(\boldsymbol{G}\phi)(\theta) = \int_{\mathbb{R}_+} G(\theta, \theta')\mu(d\theta')\phi(\theta').$$

To fix ideas we set $g_0 = \beta = \gamma \equiv 0$.

$$X_t^\alpha = \int_{\mathbb{R}_+} \mu(d\theta) Y_t^\alpha(\theta)$$

Controlled process Y^α

$$dY_t^\alpha = (A^{mr} Y_t^\alpha + \boldsymbol{B} Y_t^\alpha + C\alpha_t)\, dt + (\boldsymbol{D} Y_t^\alpha + F\alpha_t)\, dW_t, \quad Y_0^\alpha = 0,$$

Cost functional

$$J(\alpha) = \mathbb{E}\left[\int_0^T \left(\langle Y_s^\alpha, \boldsymbol{Q} Y_s^\alpha \rangle_\mu + \alpha_s^\top N \alpha_s\right) ds\right],$$

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Heuristic derivation

LQ structure of the problem suggests a value function of the form:

$$V_t^{\alpha^*} = \langle Y_t^{\alpha^*}, \boldsymbol{\Gamma}_t Y_t^{\alpha^*} \rangle_\mu,$$

with an optimal feedback control α^* satisfying

$$\alpha_t^* = -\left(N + F^* \boldsymbol{\Gamma}_t F\right)^{-1} \left(C^* \boldsymbol{\Gamma}_t + F^* \boldsymbol{\Gamma}_t \boldsymbol{D}\right) Y_t^{\alpha^*},$$

where $\boldsymbol{\Gamma}_t$ is an auto-adjoint operator from $L^1(\mu)$ into $L^\infty(\mu^\top)$, and solves the operator Riccati equation:

$$\begin{cases} \boldsymbol{\Gamma}_T & = & \mathbf{0} \\ \dot{\boldsymbol{\Gamma}}_t & = & -\boldsymbol{\Gamma}_t A^{mr} - (\boldsymbol{\Gamma}_t A^{mr})^* - \boldsymbol{Q} - \boldsymbol{D}^* \boldsymbol{\Gamma}_t \boldsymbol{D} - \boldsymbol{B}^* \boldsymbol{\Gamma}_t - (\boldsymbol{B}^* \boldsymbol{\Gamma}_t)^* \\ & & + \left(C^* \boldsymbol{\Gamma}_t + F^* \boldsymbol{\Gamma}_t \boldsymbol{D}\right)^* \left(N + F^* \boldsymbol{\Gamma}_t F\right)^{-1} \left(C^* \boldsymbol{\Gamma}_t + F^* \boldsymbol{\Gamma}_t \boldsymbol{D}\right) \end{cases}$$

Verification argument

$$X_t^\alpha = \int_{\mathbb{R}_+} \mu(d\theta) Y_t^\alpha(\theta)$$

$$dY_t^\alpha(\theta) = \left(-\theta Y_t^\alpha(\theta) + B \int_{\mathbb{R}_+} \mu(d\theta') Y_t^\alpha(\theta') + C\alpha_t \right) dt + \left(D \int_{\mathbb{R}_+} \mu(d\theta') Y_t^\alpha(\theta') + F\alpha_t \right) dW_t, \quad Y_0^\alpha(\theta) = 0,$$

Ansatz:

$$V_t^\alpha = \langle Y_t^\alpha, \mathbf{\Gamma}_t Y_t^\alpha \rangle_\mu = \int_{\mathbb{R}_+^2} Y_t^\alpha(\theta)^\top \mu(d\theta)^\top \Gamma_t(\theta,\tau) \mu(d\tau) Y_t^\alpha(\tau)$$

Define

$$S_t^\alpha := V_t^\alpha + \int_0^t \left(\langle Y_s^\alpha, \boldsymbol{Q} Y_s^\alpha \rangle_\mu + \alpha_s^\top N \alpha_s \right) ds$$

Strategy (as previously): Prove that S_t^α is a submartingale, by completion of squares technique, and make the optimal control α^* appear...

Verification argument

$\Rightarrow$ Since $Y_t(\theta)$ semimartingale, apply Itô θ by θ on

$$t \to Y_t^{\alpha}(\theta)\Gamma_t(\theta,\tau)Y_t(\tau).$$

After completion of squares: Vanishing quadratic term yields the Riccati equation for Γ

$$\Gamma_t(\theta,\tau) = \int_t^T e^{-(\theta+\tau)(s-t)}\mathcal{R}_1(\Gamma_s)(\theta,\tau)ds, \quad \mu\otimes\mu - a.e.$$

$$\mathcal{R}_1(\Gamma)(\theta,\tau) = Q + D^\top \int_{\mathbb{R}_+^2} \mu(d\theta')^\top \Gamma(\theta',\tau')\mu(d\tau')D + B^\top \int_{\mathbb{R}_+} \mu(d\theta')^\top \Gamma(\theta',\tau) + \int_{\mathbb{R}_+} \Gamma(\theta,\tau')\mu(d\tau')B - S(\Gamma)(\theta)^\top \hat{N}^{-1}(\Gamma)S(\Gamma)(\tau)$$

Verification argument

Verification result

Assume that

1. There exists a global solution $\Gamma \in C([0,T], L^1(\mu \otimes \mu))$ to the Riccati:

$$\Gamma_t(\theta,\tau) = \int_t^T e^{-(\theta+\tau)(s-t)} \mathcal{R}_1(\Gamma_s)(\theta,\tau)ds$$

2. There exists an admissible control α^* satisfying

$$\alpha_t^* = -\hat{N}(\Gamma_t)^{-1} \int_{\mathbb{R}_+} S(\Gamma_t)(\theta)\mu(d\theta) Y_t^{\alpha^*}(\theta)$$

Then, α^* is an optimal control and $V_t^{\alpha^*} = \langle Y_t^{\alpha^*}, \mathbf{\Gamma}_t Y_t^{\alpha^*} \rangle_\mu$ is the value function of the problem:

$$V_t^{\alpha^*} = \inf_\alpha J_t(\alpha)$$

Existence Riccati

Riccati equation

Assume

$$Q \in \mathbb{S}_+^d, \quad N - \lambda I_m \in \mathbb{S}_+^m,$$

for some $\lambda > 0$. Then, there exists a unique solution $\Gamma \in C([0,T], L^1(\mu \otimes \mu))$ to the Riccati equation such that for all $t \leq T$

$$\Gamma_t(\theta,\tau) = \Gamma_t(\tau,\theta)^\top, \quad \mu \otimes \mu - a.e.,$$

and

$$\int_{\mathbb{R}_+} \phi(\theta)^\top \mu(d\theta)\Gamma_t(\theta,\tau)\mu(d\tau)\phi(\tau) \geq 0, \quad \phi \in L^\infty(\mu).$$

Furthermore, there exists some positive constant $M > 0$ such that

$$\int_{\mathbb{R}_+} |\mu|(d\tau)|\Gamma_t(\theta,\tau)| \leq M, \quad \mu - a.e., \quad 0 \leq t \leq T.$$

Approximation of LQ Volterra

Intuition for the *approximation*:

1. $K(t) = \int_{\mathbb{R}_+} e^{-\theta t} \mu(d\theta)$,
2. Approximate μ by $\mu^n = \sum_{i=1}^n c_i \delta_{\theta_i}$,
3. $K^n(t) := \int_{\mathbb{R}_+} e^{-\theta t} \mu^n(d\theta) = \sum_{i=1}^n c_i e^{-\theta_i t} \to K(t)$,
4.

$$X_t^{n,\alpha} = g_0^n(t) + \int_0^t K^n(t-s) dZ_s^{n,\alpha}$$
$$\downarrow$$
$$X_t^{\alpha} = g_0(t) + \int_0^t K(t-s) dZ_s^{\alpha}.$$

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$$X_t^{\alpha} \stackrel{\text{}}{=} g_0(t) + \int_0^t K(t-s)dZ_s^{\alpha}.$$

Approximation of LQ Volterra

By substituting (K, g_0) with (K^n, g_0^n), the approximating problem reads

$$V_0^n = \inf_{\alpha \in \mathcal{A}} J^n(\alpha)$$

where

$$J^n(\alpha) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T \left((X_s^{n,\alpha})^\top Q X_s^{n,\alpha} + \alpha_s^\top N \alpha_s\right) ds\right]$$

Main result 2: Stability

Assume $(N - \lambda I_m) \in \mathbb{S}_+^m$ and that Q is invertible. Denote by V^* and V^{n*} the respective optimal value processes for the respective inputs (g_0, K) and (g_0^n, K^n), for $n \geq 1$. If

$$\|K^n - K\|_{L^2(0,T)} \to 0 \quad \text{and} \quad \|g_0^n - g_0\|_{L^2(0,T)} \to 0, \quad \text{as } n \to \infty,$$

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Choice for $(K^n)_n$, fractional case

Recall that $K(t) = \int_{\mathbb{R}_+} e^{-\theta t}\mu(d\theta)$. Set $K^n(t) = \sum_{i=1}^n c_i^n e^{-\theta_i^n t}$ with

$$c_i^n = \int_{\eta_{i-1}^n}^{\eta_i^n} \mu(d\theta) \quad \text{and} \quad \theta_i^n = \frac{1}{c_i^n}\int_{\eta_{i-1}^n}^{\eta_i^n} \theta\mu(d\theta),$$

for some partition $0 \leq \eta_1^n \leq \ldots \leq \eta_n^n$.
$\Rightarrow \|K^n - K\|_{L^2(0,T)} \to 0$.

Fractional kernel: closed form expressions

$$c_i^n = \frac{(r_n^{(1-\alpha)} - 1)}{\Gamma(\alpha)\Gamma(1-\alpha)(1-\alpha)} r_n^{(1-\alpha)i}, \quad \theta_i^n = \frac{1-\alpha}{2-\alpha}\frac{r_n^{2-\alpha} - 1}{r_n^{1-\alpha} - 1} r_n^{i-1-n/2},$$

where $\alpha := H + 1/2$, with a geometric repartition $\eta_i^n = r_n^i$ for some r_n such that

$$r_n \downarrow 1 \quad \text{and} \quad n \ln r_n \to \infty, \quad \text{as } n \to \infty.$$

See (A.J. '19, A.J. & El Euch '19)

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Practical relevance

Set $d = d' = m = 1$ ($g_0 \equiv 0$).

$$X_t^{n,\alpha} = \int_{\mathbb{R}_+} \mu^n(d\theta) Y_t^\alpha(\theta) = \sum_{i=1}^n c_i^n Y_t^\alpha(\theta_i^n)$$

where, after setting $Y^{n,i,\alpha} := Y^\alpha(\theta_i^n)$,

$$dY_t^{n,i,\alpha} = \left(-\theta_i^n Y_t^{n,i,\alpha} + B \sum_{j=1}^n c_j^n Y_t^{n,j,\alpha} + C\alpha_t \right) dt$$
$$+ \left(D \sum_{j=1}^n c_j^n Y_t^{n,j,\alpha} + F\alpha_t \right) dW_t, \quad Y_0^{n,i,\alpha} = 0, \quad i = 1, \ldots, n,$$

- $(Y^{n,i,\alpha})_{1 \le i \le n}$ is a conventional Markovian LQ problem in $\mathbb{R}^n$.
- Riccati equation in $L^1(\mu^n)$ reduces to the standard $n \times n$-matrix Riccati equation which can be solved numerically.

Stability result $\Rightarrow$ Approximation of LQ Volterra problem by conventional Markovian LQ problems in finite dimension.

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Wrap-up

- Martingale verification argument as in conventional case.
- Infinite dimensional control in Banach space: known results in Hilbert spaces cannot be applied
- Generic existence and uniqueness results for Riccati equations in $L^1(\mu \otimes \mu)$,
- LQ Volterra problems can be identified/approximated with **conventional Markovian LQ problems**,

Questions

For the more details on what was presented :

- **Linear–Quadratic control for a class of stochastic Volterra equations: solvability and approximation**, 2019, Abi Jaber, Miller, Pham,
- **Integral operator Riccati equations arising in stochastic Volterra control problems**, 2019, Abi Jaber, Miller, Pham.

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