Markowitz portfolio selection for multivariate affine and quadratic Volterra models

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Joint work with
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Motivation

The Markowitz (1952) mean-variance portfolio selection problem is the cornerstone of modern portfolio allocation theory. Investment decisions rules are made according to a tradeoff between return and risk. The use of Markowitz efficient portfolio strategies in the financial industry has become quite popular mainly due to its natural and intuitive formulation.

$$\min_{\substack{\mathbb{E}(X^\pi)=m \\ \pi\in\text{Admissible strategies}}} \mathbb{V}(X^\pi)$$

In the direction of more realistic modeling of asset prices, it is now well-established since the seminal paper by Gatheral et al. (2018) that volatility is rough, modeled by Fbm with small Hurst parameter $H \approx 0.1$. In an empirical study, Glasserman & He (2020) observed that the buy rough sell smooth was yielding superior returns.

Question: How is the investment strategy influenced by the H_i's ? Can we recover the buy rough sell smooth strategy ? Tractable numerics ?

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An example with two stocks

Consider a financial market on $[0, T]$ with two stocks S^1 and S^2 :

$$\begin{cases} dS_t^i &= S_t^i \left(\theta (Y_t^i)^2 dt + Y_t^i d\tilde{B}_t^i\right), \\ Y_t^i &= Y_0 + \int_0^t (t-s)^{H_i - 1/2} \eta_i dW_s^i, \quad i = 1, 2, \end{cases}$$

with $0 < H_1 < H_2 \leq 1/2$ and

$$\tilde{B}^1 = B^1, \quad \tilde{B}^2 = \rho B^1 + \sqrt{1-\rho^2} B^2, \quad W^i = c_i \tilde{B}^i + \sqrt{1-c_i^2} \tilde{B}^{i,\perp},$$

where $(B^1, B^2, B^{1,\perp}, B^{2,\perp})$ is a four dimensional Brownian motion.

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Literature review

The research on portfolio optimization and multivariate rough models is still little developed but has gained an increasing attention :

- Fractional OU environment (1d)+ power utility : Fouque & Hu (2018)
- Multidimensional setting (no control) : Abi Jaber (2019), Cuchiero & Teichman (2019), Rosenbaum & Tomas (2019)
- Rough Heston (1d) + power utility : Bäuerle & Demestre (2020)
- Rough Heston (1d) + Markowitz : Han & Wong (2020)

Challenges and Limitations:

- Passing to the multidimensional case + Tractable numerics

In our paper: We solve the Markowitz problem both in the multivariate affine (Rough Heston) and quadratic (Stein-Stein) models and study numerically the quadratic case. We recover the buy rough sell smooth strategy when $\rho > 0$ and exhibit a transition from $T \ll 1$ to $T \gg 1$.

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Motivation

Consider a financial market on $[0, T]$ a non–risky asset $S^0 = 1$, and d risky assets with dynamics

$$dS_t = \text{diag}(S_t)\Big[\big(\sigma_t \lambda_t\big)dt + \sigma_t dB_t\Big].$$

- σ : $\mathbb{R}^{d\times d}$- valued stochastic volatility process,
- λ : $\mathbb{R}^{d}$-valued stochastic market price of risk ($\approx \frac{€}{risk}$)

The model

Let

- $N = (N^1, ..., N^d)$: numbers of shares bought in the risky assets $(S^1, ..., S^d)$,
- $\pi = (\pi^1, ..., \pi^d) = N^\top \mathrm{diag}(S)$: amounts invested in the risky assets $(S^1, ..., S^d)$ ($\approx$ €),
- $\alpha = \sigma^\top \pi$

Then, the dynamics of the wealth $X_t = N_t^\top S_t + \pi_t^0$ of the self-financing portfolio is given by

$$\begin{aligned} dX_t &= N_t^\top dS_t \\ &= N_t^\top \mathrm{diag}(S_t)\left[(\sigma_t \lambda_t) dt + \sigma_t dB_t\right] \\ &= \alpha_t^\top (\lambda_t dt + dB_t), \quad X_0 = x_0 \in \mathbb{R}. \end{aligned}$$

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As a result, our problem reduces to

$$V(m) = \min_{\substack{\mathbb{E}(X_T)=m \\ \alpha\in\mathcal{A}}} \mathbb{V}(X_T),$$

under the constrain

$$dX_t = \alpha_t^\top \left(\lambda_t dt + dB_t\right), \quad X_0 = x_0 \in \mathbb{R}.$$

(Quadratic) Rough volatility assumption :

- $\lambda = \Theta Y$, ($\sigma_{ij} = \gamma_{ij}^\top Y$, $dS_t = \mathrm{diag}(S_t)\left[\left(\sigma_t\lambda_t\right)dt + \sigma_t dB_t\right]$)
- $Y_t = g_0(t) + \int_0^t K(t,s)\eta dW_s \in \mathbb{R}^N$, Volterra OU-process,
- W : N-dimensional BM correlated to B via $W_t^k = C_k^\top B_t + \sqrt{1 - C_k^\top C_k} B_t^{\perp,k}, \quad k = 1, \dots, N.$
- K : a general non convolution kernel in L^2. Today we only present the multivariate quadratic case, we refer to our paper for the affine (Heston) setting.

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$$\begin{cases} \min_{\alpha \in \mathcal{A}} & \mathbb{V}(X), \qquad \mathbb{E}(X_T) = m, \\ dX_t = & \alpha_t^\top \left(\lambda_t dt + dB_t\right), \quad X_0 = x_0 \in \mathbb{R}. \end{cases}$$

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Verification theorem

Assume that there exists a solution triplet (Γ, Z^1, Z^2) to the Riccati *BSDE**

$$\begin{cases} d\Gamma_t &= \Gamma_t\Big[\big|\lambda_t + Z_t^1 + CZ_t^2\big|^2 dt + \big(Z_t^1\big)^\top dB_t + \big(Z_t^2\big)^\top dW_t\Big], \\ \Gamma_T &= 1, \end{cases}$$

Then, the optimal investment strategy is given by

$$\alpha_t^* = \big(\lambda_t + Z_t^1 + CZ_t^2\big)\big(\xi^* - X_t^*\big), \qquad \xi^* = \frac{m - \Gamma_0 x_0}{1 - \Gamma_0},$$

and the value of the optimal wealth process is

$$V(m) = \mathbb{V}(X_T^*) = \Gamma_0 \frac{\big|x_0 - m\big|^2}{1 - \Gamma_0}.$$

* with some additional hypothesis on $\mathbb{E}[\exp(\int_0^T |\lambda_s|^2 ds)]$ and Γ.

Sketch of proof

It is well-known that the Markowitz problem is equivalent to the following max-min problem,

$$\min_{\substack{\mathbb{E}(X_T)=m \\ \alpha\in\mathcal{A}}} \mathbb{V}(X_T^\alpha) = \max_{\eta\in\mathbb{R}} \min_{\alpha\in\mathcal{A}} \left\{ \mathbb{E}\left[\left| X_T^\alpha - (m-\eta) \right|^2 \right] - \eta^2 \right\}.$$

Thus, two optimization problems have to be solved :

1. the internal minimization problem over $\alpha \in \mathcal{A} \rightarrow$ stochastic LQ problem,
2. the external maximization problem over $\eta \in \mathbb{R} \rightarrow$ simple minimization of a 2nd degree polynomial.

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Some remarks

$$\alpha_t^* = \left(\lambda_t + Z_t^1 + CZ_t^2\right)\left(\xi^* - X_t^*\right)$$

- It can be proved that $\xi^* - X_t^* \geq 0$ on [0,T].
- Consequently, to grasp the effect of the roughness's of stocks upon the investment strategy, one needs to understand its effect on Z^1 and Z^2.
- We will derive explicit formulae for the the triplets (Γ, Z^1, Z^2).

Verification theorem

	Random coef.	Unbounded coef.	degenerate σ	Incomplete market
Lim & Zhou (2002)	✓	✗	✗	✗
Lim (2004)	✓	✗	✗	✓
Shen (2015)	✓	✓	✗	✗
Verification theorem	✓	✓	✓	✓

Table: Comparison to existing verification results for mean-variance problems.

(H1) $0 < \Gamma_0 < 1$, and $\Gamma_t > 0$,

(H2)

$$\mathbb{E}\Big[\exp\Big(a(p)\int_0^T \big(|\lambda_s|^2 + \big|Z_s^1\big|^2 + \big|Z_s^2\big|^2\big)\,ds\Big)\Big] < \infty,$$

for some $p > 2$ and a constant $a(p)$ given by

$$a(p) = \max\Big[p\,(3+|C|)\,,(8p^2-2p)\,\big(1+2|C|+|C|^2\big)\Big].$$

Some remarks

By setting $\tilde{Z}_t^i = \Gamma_t Z_t^i$, the Riccati BSDE agrees with the one in Chiu and Wong (2014, Theorem 3.1)

$$d\Gamma_t = \underbrace{\Gamma_t \underbrace{\left[\left(\left|\lambda_t + \frac{\tilde{Z}_t^1 + C\tilde{Z}_t^2}{\Gamma_t}\right|^2\right)\right]}_{\text{degree}=0}}_{\text{degree}=1} dt + \underbrace{\left(\tilde{Z}_t^1\right)^\top dB_t + \left(\tilde{Z}_t^2\right)^\top dW_t}_{\text{degree}=1},$$

which is the Riccati BSDE one naturally encounters when solving the LQ control problem.

- This observation on the degree of the equation allows us to avoid the **Martingale distortion transformation** ($\Gamma^a, a \in \mathbb{R}$) which only works in dimension 1 ! See Fouque & Hu (2018).

Understanding (Γ, Z^1, Z^2)

Recall that we need to solve

$$\begin{cases} d\Gamma_t &= \Gamma_t\Big[\left|\lambda_t + Z_t^1 + CZ_t^2\right|^2 dt + \left(Z_t^1\right)^\top dB_t + \left(Z_t^2\right)^\top dW_t\Big], \\ \Gamma_T &= 1, \end{cases}$$

Key idea : 0bserve that if such solution exists, then, it admits the following representation as a Laplace transform:

$$\Gamma_t = \mathbb{E}\Big[\exp\Big(-\int_t^T \left(\left|\lambda_s + Z_s^1 + CZ_s^2\right|^2\right) ds\Big) \,\Big|\, \mathcal{F}_t\Big], \quad 0 \le t \le T.$$

- If λ deterministic $\implies \Gamma_t = e^{-\int_t^T \lambda_s ds}, Z^1 = Z^2 = 0.$

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Recall that we need to solve

$$\begin{cases} d\Gamma_t &= \Gamma_t \Big[\big|\lambda_t + Z_t^1 + C Z_t^2\big|^2 dt + \big(Z_t^1\big)^\top dB_t + \big(Z_t^2\big)^\top dW_t \Big], \\ \Gamma_T &= 1, \end{cases}$$

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Heuristic

$$\Gamma_t \;=\; \mathbb{E}\Big[\exp\Big(-\underbrace{\int_t^T \big(\big|\lambda_s + Z_s^1 + CZ_s^2\big|^2\big)ds}_{\approx\text{squared gaussian}}\Big) \;\Big|\; \mathcal{F}_t\Big].$$

Or, if $G \sim N(\mu, \Sigma)$ in $\mathbb{R}^n$, then

$$\mathbb{E}\left(\exp(-u|G|^2)\right) = \frac{\exp\left(-u\left(\mu^\top (I_n + 2\Sigma u)^{-1}\mu\right)\right)}{det(I_n + 2\Sigma u)^{1/2}}$$

Idea : Make the approximation, see Abi Jaber (2019) :

$$\int_t^T G_s^2 ds \approx n^{-1}\sum_{i=1}^n G_{i/n}^2 \sim |N(\mu_n, \Sigma_n)|^2$$

Heuristic

A a result, we expect

$$\Gamma_t = \mathbb{E}\Big[\exp\Big(-\int_t^T \big(\big|\lambda_s + Z_s^1 + CZ_s^2\big|^2\big)ds\Big)\ \Big|\ \mathcal{F}_t\Big]$$
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Questions :

- To what limit do these object of length n converge as $n \to \infty$?

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As a result we expect

$$\Gamma_t = \mathbb{E}\Big[\exp\Big(-\int_t^T \big(\big| \lambda_s + Z_s^1 + C Z_s^2 \big|^2 \big) ds \Big) \Big| \underbrace{\mathcal{F}_t}_{\text{randomness over } [0,t]} \Big]$$

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1. To what limit do these objects of length n converge as $n \to \infty$?
2. What should play the role of μ_n in our setting ?

Heuristic

Questions :

1. To what limit do these object of length n converge as $n \to \infty$?
   - As $n \to \infty$, big matrices converge to operators. A natural infinite dimensional space appears : $L^2([0, T])$.
2. What should play the role of μ_n in our setting ?
   - With respect to what information Y is markovian ?
     $\to g_t(s) = \mathbb{E}\left[Y_s \mid \mathcal{F}_t\right]$, $s \geq t$.

Heuristic

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$$\approx \exp(\phi_t + \langle g_t, \boldsymbol{\Psi}_t g_t\rangle_{L^2})$$

- This limit argument will guide us to approximate the infinite dimensional object $\boldsymbol{\Psi}$. However, we do not use this argument throughout our paper and work all the way long in the infinite dimensional setting.

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The setting

- Let $\langle \cdot, \cdot \rangle_{L^2}$ be inner product on $L^2\left([0,T], \mathbb{R}^N\right)$ that is $\langle f, g \rangle_{L^2} = \int_0^T f(s)^\top g(s) ds$.
- $\forall K \in L^2\left([0,T]^2, \mathbb{R}^{N \times N}\right)$, we denote by $\boldsymbol{K}$ the integral operator induced :$(\boldsymbol{K} g)(s) = \int_0^T K(s,u) g(u) du$.
- $\boldsymbol{K}$ is said to be positive if $\langle f, \boldsymbol{K} f \rangle_{L^2} \geq 0$.

Riccati operator

Inspired by Abi Jaber (2019), we provide an explicit solution to the Riccati BSDE, in terms of the following family of linear operators acting on $L^2\left([0,T],\mathbb{R}^N\right)$:

$$\boldsymbol{\Psi}_t = -\left(\mathrm{Id} - \hat{\boldsymbol{K}}\right)^{-*}\Theta^\top\left(\mathrm{Id} + 2\Theta\tilde{\boldsymbol{\Sigma}}_t\Theta^\top\right)^{-1}\Theta\left(\mathrm{Id} - \hat{\boldsymbol{K}}\right)^{-1}, \quad 0 \le t \le T,$$

where

- $\hat{\boldsymbol{K}}$ is the integral operator induced by the kernel $\hat{K} = -2K(\eta C^\top \Theta)$
- $\tilde{\boldsymbol{\Sigma}}_t = (\mathrm{Id} - \hat{\boldsymbol{K}})^{-1}\boldsymbol{\Sigma}_t(\mathrm{Id} - \hat{\boldsymbol{K}})^{-*}$
- $\boldsymbol{\Sigma}_t$ defined as the integral operator associated to the kernel

$$\Sigma_t(s,u) = \int_t^{s\wedge u} K(s,z)\eta(U - 2C^\top C)\eta^\top K(u,z)^\top dz, \qquad t \in [0,T],$$

where $U = \frac{d\langle W\rangle_t}{dt} = \left(1_{i=j} + 1_{i\neq j}(C_i)^\top C_j\right)_{1\le i,j\le N}$.

Riccati operator

Inspired by Abi Jaber (2019), we provide an explicit solution to the Riccati BSDE, in terms of the following family of linear operators acting on $L^2\left([0,T],\mathbb{R}^N\right)$:

$$\boldsymbol{\Psi}_t = -\left(\mathrm{Id} - \hat{\boldsymbol{K}}\right)^{-*}\Theta^\top\left(\mathrm{Id} + 2\Theta\tilde{\boldsymbol{\Sigma}}_t\Theta^\top\right)^{-1}\Theta\left(\mathrm{Id} - \hat{\boldsymbol{K}}\right)^{-1}, \quad 0 \le t \le T,$$

where

- $\hat{\boldsymbol{K}}$ is the integral operator induced by the kernel $\hat{K} = -2K(\eta C^\top\Theta)$
- $\tilde{\boldsymbol{\Sigma}}_t = (\mathrm{Id} - \hat{\boldsymbol{K}})^{-1}\boldsymbol{\Sigma}_t(\mathrm{Id} - \hat{\boldsymbol{K}})^{-*}$
- $\boldsymbol{\Sigma}_t$ defined as the integral operator associated to the kernel

$$\Sigma_t(s,u) = \int_t^{s\wedge u} K(s,z)\eta\big(U - 2C^\top C\big)\eta^\top K(u,z)^\top dz, \qquad t \in [0,T],$$

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$$\boldsymbol{\Psi}_t = -\left(\mathrm{Id}-\hat{\boldsymbol{K}}\right)^{-*}\underbrace{\Theta^\top\left(\mathrm{Id}+2\Theta\tilde{\boldsymbol{\Sigma}}_t\Theta^\top\right)^{-1}\Theta}_{\approx \lim_{n\to\infty} u(I_n+2\Sigma_n u)^{-1}}\left(\mathrm{Id}-\hat{\boldsymbol{K}}\right)^{-1}, \quad t\in[0,T],$$

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$$\approx Cov(g_t(s), g_t(u))$$

Riccati Operator

1. $t \mapsto \boldsymbol{\Psi}_t$ is strongly differentiable and satisfies the operator Riccati equation

$$\dot{\boldsymbol{\Psi}}_t = 2\boldsymbol{\Psi}_t \dot{\boldsymbol{\Sigma}}_t \boldsymbol{\Psi}_t, \qquad t \in [0, T]$$

$$\boldsymbol{\Psi}_T = -\left(\mathrm{Id} - \hat{\boldsymbol{K}}\right)^{-*} \Theta^\top \Theta \left(\mathrm{Id} - \hat{\boldsymbol{K}}\right)^{-1}$$

   where $\dot{\boldsymbol{\Sigma}}_t$ is the strong derivative of $t \mapsto \boldsymbol{\Sigma}_t$.
2. $\forall f \in L^2 \qquad (\boldsymbol{\Psi}_t f 1_t)(t) = (-\Theta^\top \Theta \mathrm{Id} + \hat{\boldsymbol{K}}^* \boldsymbol{\Psi}_t)(f)(t)$
3. For any $t \in [0, T]$, $(\Theta^\top \Theta \mathrm{Id} + \boldsymbol{\Psi}_t)$ is an integral operator.

$^* 1_t : s \mapsto \mathbf{1}_{t \leq s}$.

Deriving the solution

Riccati BSDE - Riccati operator - Forward process

Then, the process $\left(\Gamma, Z^1, Z^2\right)$ defined by

$$\begin{cases} \Gamma_t &= \exp\left(\phi_t + \langle g_t, \boldsymbol{\Psi}_t g_t \rangle_{L^2}\right), \\ Z_t^1 &= 0, \\ Z_t^2 &= 2\big((\boldsymbol{\Psi}_t \boldsymbol{K}\eta)^* g_t\big)(t), \end{cases}$$

is solution to the Riccati BSDE, where $\Phi_t = \ln(det(\boldsymbol{\Psi}_t \boldsymbol{\Lambda}_t))$.

Optimal control

Consequently, the optimal control in the Quadratic model is of the form

$$\alpha_t^* = \left(\left(\Theta + 2C\left[\boldsymbol{\Psi}_t \boldsymbol{K}\eta\right]^*\right) g_t\right)(t)\left(\xi^* - X_t^{\alpha^*}\right),$$

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$$\alpha_t^* = \underbrace{\Big(\big(\Theta + 2C\left[\mathbf{\Psi}_t \boldsymbol{K}\eta\right]^*\big) g_t\Big)(t)}_{\text{Numerically tractable with simple linear algebra !}} \left(\xi^* - X_t^{\alpha^*}\right),$$

Link with the classical setting

Set $K(t,s) = I_N 1_{s \leq t}$,

Lemma - From L^2 to $\mathbb{R}^N$

Define $P_t = \int_t^T (\boldsymbol{\Psi}_t \mathbf{1}_t)(s) ds$ with $\mathbf{1}_t : (s) \mapsto (1_{t \leq s}, \ldots, 1_{t \leq s})^\top$. Then $t \to P_t \in \mathbb{R}^N$ is solution to a classical Riccati equation :

$$\dot{P}_t = \Theta^\top \Theta + P_t M + M^\top P_t - P_t Q P_t, \qquad P_T = 0.$$

Link to Chiu & Wong (2014)

Then, the solution to the Riccati BSDE can be re-written in the form

$$\Gamma_t = \exp\left(\phi_t + Y_t^\top P_t Y_t\right), \quad \text{and} \quad Z_t^2 = 2(D\eta)^\top P_t Y_t,$$

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- As we'll see, $\boldsymbol{\Psi}$ can be easily computed with simple linear algebra. Does this open a new way of computing classical Riccati equation ?

Numerics

We share the code on a notebook.

Wrap-up

- In our paper we solve the Markowitz portfolio allocation problem both in the multidimensional Heston and Stein Stein setting.
- The infinite dimensional nature of the solution is numerically tractable with simple linear algebra.
- We recover the buy rough & sell smooth strategy exhibited in Glasserman (2020) when stocks are positively correlated.
- We observe an interesting transition from short $T \ll 1$ to long $T \gg 1$ time scale ($\mathbb{V} \approx t^{2H}$, or $t^{2H_1} > t^{2H_2}$ if $t < 1$, then $t^{2H_1} < t^{2H_2}$ for $t > 1$).

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Questions

For the more details on what was presented :

- **Markowitz portfolio selection for multivariate affine and quadratic Volterra models**, 2020, Abi Jaber, Miller, Pham.

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